Berry-Esseen's bound and Cramér's large deviation expansion for a supercritical branching process with immigration in a random environment

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CONTENTS

Outline

BPRE

Branching processes in random environments (BPRE) have been studied by many authors, since the introduction of the model by Smith and Wilkinson (1969) and the fundamental work of Athreya and Karlin (1971). Important progress has been made in recent years, see e.g.:

- Critical and subcritical cases, survival probability and conditional limit theorems: Vatutin & Dyakonova (2020, 2018), Vatutin & Wachtel (2018), Le Page, Peigné & Pham (2018) in the multi-type case, Afanasyev, Böinghoff, Kersting & Vatutin (2014, 2012), Afanasyev, Geiger, Kersting & Vatutin (2005) in the single type case.
- Supercritical case, large deviations: Buraczewski & Dyszewski (2020), Grama, Liu & Miqueu (2017), Bansaye & Böinghoff (2014, 2013, 2011), Huang & Liu (2012), Bansaye & Berestycki (2009).

See also the book by Kersting & Vatutin (2017) and many ref. therein.

BPIRE

Branching processes with immigration in random environments (BPIRE) have been less studied, but deserve our attention due to a number of applications in various fields. For example:

- Kesten, Kozlov and Spitzer (1975), Key (1984), Hong and Zhang(2016) used a BPIRE to give limit laws for a random walk in a random environment;
- Bansaye (2009) studied a model of cell contamination by investigating a BPIRE;
- Vatutin (2011) considered a BPIRE to study polling systems with random regimes of service.

With the immigration, some properties of the branching process remain the same, while some others become different.

Objective

We consider a supercritical branching process (Z_n) with immigration in an i.i.d. environment $\xi = (\xi_n)$, with

$$\mu = \mathbb{E} \log m_0 > 0, \quad \sigma^2 = Var \log m_0 \in (0, \infty),$$

where $m_0 = \sum k p_k(\xi_0)$ is the expected value of the offspring distribution $\{p_k(\xi_0) : k \geq 0\}$ at time 0, given the environment. For simplicity, assume $p_0(\xi_0) = 0$. We are interested in:

1. the absolute error of the Gaussian approximation:

$$\sup_{x \in \mathbb{R}} \left| \mathbb{P}\left(\frac{\log Z_n - n\mu}{\sqrt{n}\sigma} \leq x \right) - \Phi(x) \right| \approx ? \quad \Phi(x) = \int_{-\infty}^{x} \frac{e^{-t^2/2}}{\sqrt{2\pi}} dt;$$

2. the relative error of the Gaussian approximation:

$$\frac{\mathbb{P}\left(\frac{\log Z_n - n\mu}{\sigma\sqrt{n}} > x \right)}{1 - \Phi(x)} \sim ?$$

Outline

Description of BPIRE

Let $\xi = (\xi_0, \xi_1, \xi_2, \cdots)$ be a sequence of independent and identically distributed random variables taking values in some space Θ indexed by time $n \in \mathbb{N} = \{0, 1, 2 \cdots\}$, which represents the random environment. Each realization of ξ_n corresponds to two probability distributions on $\mathbb{N}$. One is the offspring distribution denoted by $p(\xi_n) = \{p_k(\xi_n); k \geq 0\}$, where

$$0 \leq p_k(\xi_n) \leq 1, \ \ and \ \sum_k p_k(\xi_n) = 1.$$

The other is the immigration distribution denoted by $\hat{p}(\xi_n) = \{\hat{p}_k(\xi_n); k \geq 0\}$, where

$$0 \leq \hat{p}_k(\xi_n) \leq 1, \ \ and \ \sum_k \hat{p}_k(\xi_n) = 1.$$

Definition of a BPIRE

A branching process $(Z_n)_{n\geq 0}$ with immigration $(Y_n)_{n\geq 0}$ in the random environment ξ (BPIRE) can be defined as follows:

$$Z_0 = 1, \qquad Z_{n+1} = \sum_{i=1}^{Z_n} X_{n,i} + Y_n, \quad n = 0, 1, 2, \cdots$$

where given the environment ξ, $X_{n,i} (i = 1, 2, \cdots)$, Y_n, $n \geq 0$, are independent of each other, each $X_{n,i} (i = 1, 2, \cdots)$ has the same distribution $p(\xi_n)$, Y_n has the distribution $\hat{p}(\xi_n)$.

Quenched law and annealed law

Let $(\Gamma, \mathbb{P}_\xi)$ be the probability space under which the process is defined when the environment ξ is given. As usual, $\mathbb{P}_\xi$ is called quenched law.

The total probability space can be formulated as the product space $(\Gamma \times \Theta^{\mathbb{N}}, \mathbb{P})$, where $\mathbb{P}(dx, d\xi) = \mathbb{P}_\xi(dx)\tau(d\xi)$, τ is the law of the environment ξ. The total probability $\mathbb{P}$ is usually called annealed law.

The quenched law $\mathbb{P}_\xi$ may be considered to be the conditional probability of the annealed law $\mathbb{P}$ given ξ. The expectation with respect to $\mathbb{P}_\xi$ (resp. $\mathbb{P}$) will be denoted by $\mathbb{E}_\xi$ (resp. $\mathbb{E}$).

Outline

Limit theorems on a BPIRE

The following limit theorems have been proved in Wang and Liu (2017) for a BPIRE under suitable conditions (with $p_0(\xi_0) = 0$ a.s.):

1. Law of large numbers (LLN): $\frac{\log Z_n}{n} \to E \log m_0$ a.s.
2. Central limit theorem (CLT): with $\Phi(x) = \int_{-\infty}^{x} \frac{e^{-t^2/2}}{\sqrt{2\pi}} dt$,
$$\sup_{x\in\mathbb{R}} \left| P\left(\frac{\log Z_n - nE \log m_0}{\sigma\sqrt{n}} \le x \right) - \Phi(x) \right| \to 0$$
3. Large Deviation Principle (LDP), which gives an equivalent for
$$\log P\left(\frac{\log Z_n - nE \log m_0}{n} > \varepsilon\right), \text{ for fixed } \varepsilon > 0.$$
4. Moderate Deviation Principle (MDP), which gives an equivalent of
$$\log P\left(\frac{\log Z_n - nE \log m_0}{a_n} > \varepsilon\right), \text{ where } \frac{a_n}{n} \to 0, \frac{a_n}{\sqrt{n}} \to \infty.$$

Berry-Esseen's bound and Cramér's MD expansion

Our main objective is to show Berry-Esseen's bound and Cramér's moderate deviation expansion for $\log Z_n$: under suitable conditions, we will prove

1. Berry-Esseen's bound, which gives the rate of convergence in the central limit theorem, the absolute error in the Gaussian approximation: with $\Phi(x) = \int_{-\infty}^{x} \frac{e^{-t^2/2}}{\sqrt{2\pi}} dt$,
$$\sup_{x\in\mathbb{R}} \left| \mathbb{P}\left(\frac{\log Z_n - nE\log m_0}{\sigma\sqrt{n}} \leq x \right) - \Phi(x) \right| \leq \frac{C}{\sqrt{n}}$$
2. Cramér's moderate deviation expansion for $\log Z_n$, which improves the MDP, and gives an asymptotic expression of the relative error of the Gaussian approximation:
$$\frac{\mathbb{P}\left(\frac{\log Z_n - nE\log m_0}{\sigma\sqrt{n}} > x \right)}{1-\Phi(x)} \sim \ldots, \qquad 0 \leq x = o(\sqrt{n}).$$

Berry-Esseen's bound

Let $m_0 = \sum_k kp_k(\xi_0)$. Assume that

$$p_0(\xi_0) = 0 \text{ a.s.}, \quad \mu = \mathbb{E}\log m_0 \in (0,\infty), \quad \sigma^2 = var\,(\log m_0) \in (0,\infty).$$

A1. $\exists\delta \in (0,1]$ such that $\mathbb{E}|\log m_0|^{2+\delta} < \infty$.
A2. $\exists p > 1$ such that $\mathbb{E}\left(\frac{X_0}{m_0}\right)^p < \infty$ and $\mathbb{E}\left(\frac{Y_0}{m_0}\right)^p < \infty$, where X_0 has the offspring distribution $\{p_k(\xi_0)\}$, and Y_0 has the immigration distribution $\{\hat{p}_k(\xi_0)\}$, given the environment.

Theorem 1

*Under conditions **A1** and **A2**, we have, with* $\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-t^2/2}dt$,

$$\sup_{x\in\mathbb{R}}\left|\mathbb{P}\left(\frac{\log Z_n - n\mu}{\sigma\sqrt{n}} \le x\right) - \Phi(x)\right| \le \frac{C}{n^{\delta/2}} \quad \textit{(optimal when } \delta = 1).$$

Cramér's moderate deviation expansion

Theorem 2

Assume that $\mathbb{E}m_0^{\varepsilon} < \infty$ *for some* $\varepsilon > 0$, $\mathbb{E}\frac{X_0^p}{m_0} < \infty$ *and* $\mathbb{E}\frac{Y_0^p}{m_0} < \infty$ *for some* $p > 1$. *Then for* $0 \leq x = o(\sqrt{n})$, *we have, with* $\mu = \mathbb{E}\log m_0$, *as* $n \to \infty$,

$$\frac{\mathbb{P}\left(\frac{\log Z_n - n\mu}{\sigma\sqrt{n}} > x\right)}{1 - \Phi(x)} = \exp\left\{\frac{x^3}{\sqrt{n}}\mathscr{L}\left(\frac{x}{\sqrt{n}}\right)\right\}\left[1 + O\left(\frac{1+x}{\sqrt{n}}\right)\right] \tag{3.1}$$

and

$$\frac{\mathbb{P}\left(\frac{\log Z_n - n\mu}{\sigma\sqrt{n}} < -x\right)}{\Phi(-x)} = \exp\left\{-\frac{x^3}{\sqrt{n}}\mathscr{L}\left(-\frac{x}{\sqrt{n}}\right)\right\}\left[1 + O\left(\frac{1+x}{\sqrt{n}}\right)\right], \tag{3.2}$$

where $\mathscr{L}$ *is Cramér's series of the log-Laplace transform of* $\log m_0$.

Application in the estimation of $E \log m_0$ by $\frac{\log Z_n}{n}$

1. LLN: $\frac{\log Z_n}{n}$ is a convergent estimator of the criticality para. $E \log m_0$
2. The CLT gives the Gaussian approximation of the error probability $P(|\frac{\log Z_n}{n} - E \log m_0| > \epsilon)$ with $\epsilon = \frac{\sigma x}{\sqrt{n}}$:

$$P(|\frac{\log Z_n}{n} - E \log m_0| > \frac{\sigma x}{\sqrt{n}}) \approx 2(1 - \Phi(x)) \quad \text{close to } 0$$

3. Berry-Esseen's bound gives an estimation of the absolute error in the above gaussian approximation.
4. Moderate Deviation Principle gives an estimation of

$$\log P(|\frac{\log Z_n}{n} - E \log m_0| > \varepsilon_n), \text{ where } \varepsilon_n \to 0, \sqrt{n}\varepsilon_n \to \infty$$

5. Cramér's moderate deviation expansion gives an approximation of

$$P(|\frac{\log Z_n}{n} - E \log m_0| > \varepsilon_n), \text{ where } \varepsilon_n \to 0.$$

Outline

The main idea is to compare $\log Z_n$ with the random walk

$$S_n = \log m_0 + \cdots + \log m_{n-1}$$

following Grama, Liu and Miqueu (2017), and using the decomposition

$$\log Z_n = \log W_n + S_n,$$

where $W_n := \frac{Z_n}{\Pi_n}$ with $\Pi_n = m_0 \cdots m_{n-1}$. Assume $\mathbb{E} \log m_0 > 0$.

Lemma 1 (a.s. convergence of W_n and non-degeneracy of W)

The sequence $(W_n, \mathcal{F}_n)$ is a submartingale under $\mathbb{P}_\xi$ and $\mathbb{P}$. Assume that $\mathbb{E} \log^+(Y_0/m_0) < \infty$, then the limit

$$W := \lim_{n\to\infty} W_n \text{ exists in } [0, \infty) \text{ a.s.}$$

Moreover, $\mathbb{P}(W > 0) > 0$ iff $\mathbb{E}[\frac{X_0}{m_0} \ln^+ X_0] < \infty$.

Convergence of the submartingale in $L^p(\mathbb{P}), \quad p > 1$

Lemma 2 (Convergence of (W_n) in $L^p(\mathbb{P}), \quad p > 1$)

Assume $\mathbb{P}(Y_0 = 0) < 1$. Let $p > 1$ be fixed. Then, the sequence (W_n) converges in L^p under $\mathbb{P}$ iff

$$\mathbb{E}m_0^{-p} < 1, \; \mathbb{E}\left(\frac{Y_0^p}{m_0^p}\right) < \infty \;\text{ and }\; \mathbb{E}\left(\frac{X_0^p}{m_0^p}\right) < \infty. \tag{4.1}$$

Remark

If $\mathbb{P}(Y_0 = 0) = 1$ (the usual branching process in random environment), Guivarc'h and Liu (2001) proved that $W_n \xrightarrow{L^p} W$ iff $\mathbb{E}m_0^{1-p} < 1$ and $\mathbb{E}(\frac{X_0}{m_0})^p < \infty$. Lemma 2 shows that there are indeed different behaviors caused by the immigration, compared with a branching process without immigration: the critical value for the existence of moments of W is not the same.

Exponential convergence of $\log W_n$ in L^1

The following result concerns the exponential speed of convergence of $\log W_n$ to $\log W$.

Lemma 3 (Exponential convergence of $\log W_n$ in L^1)

*Assume **A1** and **A2**. Then there exist two constants $C > 0$ and $\delta \in (0, 1)$ such that for all $n \geq 0$,*

$$\mathbb{E} |\log W_n - \log W| \leq C\delta^n. \tag{4.2}$$

Concentration inequality for $(S_n, \log Z_n)$

With the preceding lemmas we can prove a concentration inequality for the joint law of S_n and $\log Z_n$. It shows that $\log Z_n$ and S_n are similarly distributed: for all x, they are simultaneously larger than x , or simultaneously less than x, with large probability.

Lemma 4 (Concentration inequality for $(S_n, \log Z_n)$)

*Assume **A1** and **A2**. Then for all $x \in \mathbb{R}$, we have*

$$\mathbb{P}\left(\frac{\log Z_n - n\mu}{\sigma\sqrt{n}} \leq x, \frac{S_n - n\mu}{\sigma\sqrt{n}} \geq x\right) \leq \frac{C}{n^{\delta/2}} \tag{4.3}$$

and

$$\mathbb{P}\left(\frac{\log Z_n - n\mu}{\sigma\sqrt{n}} \geq x, \frac{S_n - n\mu}{\sigma\sqrt{n}} \leq x\right) \leq \frac{C}{n^{\delta/2}}. \tag{4.4}$$

Proof of the Berry - Esseen bound using the concentration inequality for $(S_n, \log Z_n)$

$$
\begin{aligned}
& \mathbb{P}\Big(\frac{\log Z_n - n\mu}{\sigma\sqrt{n}} \le x\Big) \\
= \ & \mathbb{P}\Big(\frac{\log Z_n - n\mu}{\sigma\sqrt{n}} \le x, \frac{S_n - n\mu}{\sigma\sqrt{n}} \le x\Big) + \mathbb{P}\Big(\frac{\log Z_n - n\mu}{\sigma\sqrt{n}} \le x, \frac{S_n - n\mu}{\sigma\sqrt{n}} > x\Big) \\
= \ & \mathbb{P}\left(\frac{S_n - n\mu}{\sigma\sqrt{n}} \le x\right) + \mathbb{P}\left(\frac{\log Z_n - n\mu}{\sigma\sqrt{n}} \le x, \frac{S_n - n\mu}{\sigma\sqrt{n}} > x\right) \\
& -\mathbb{P}\left(\frac{\log Z_n - n\mu}{\sigma\sqrt{n}} > x, \frac{S_n - n\mu}{\sigma\sqrt{n}} \le x\right) \\
= \ & \mathbb{P}\left(\frac{S_n - n\mu}{\sigma\sqrt{n}} \le x\right) + O\left(\frac{1}{n^{\delta/2}}\right).
\end{aligned}
$$

So Theorem 1 follows from the classical Berry-Esseen bound for S_n.

Proof of Cramér's expansion for $0 < x \leq 1$

By the Berry - Esseen bound, we have

$$\left|\mathbb{P}\left(\frac{\log Z_n - n\mu}{\sigma\sqrt{n}} > x\right) - (1 - \Phi(x))\right| \leq \frac{C}{\sqrt{n}}.$$

Dividing both sides by $1 - \Phi(x)$, we get

$$\left|\frac{\mathbb{P}\left(\frac{\log Z_n - n\mu}{\sigma\sqrt{n}} > x\right)}{1 - \Phi(x)} - 1\right| \leq \frac{C}{(1 - \Phi(1)).\sqrt{n}},$$

Therefore,

$$\frac{\mathbb{P}\left(\frac{\log Z_n - n\mu}{\sigma\sqrt{n}} > x\right)}{1 - \Phi(x)} = 1 + O\left(\frac{1}{\sqrt{n}}\right).$$

Proof of Cramér's expansion for $1 \leq x = o(\sqrt{n})$ (1)

- Measure change like Cramér's change for the associated random walk: recall that

$$\mathbb{P}(d\xi, dx) = \mathbb{P}_\xi(dx)\tau(d\xi)$$

with $\tau = \tau_0^{\otimes \mathbb{N}}$ = the law of $\xi = (\xi_0, \xi_1, \cdots)$, τ_0 = the law of ξ_0. Define the new annealed law $\mathbb{P}_\lambda$ by

$$\mathbb{P}_\lambda(d\xi, dx) = \mathbb{P}_\xi(dx)\tau_\lambda(d\xi) \tag{4.5}$$

with $\tau_\lambda = \tau_{0,\lambda}^{\otimes \mathbb{N}}$, $\tau_{0,\lambda}(dx) = \frac{m(x)^\lambda}{L(\lambda)}\tau_0(dx)$, $m(x) = \mathbb{E}(X_0|\xi_0 = x)$, $L(\lambda) = \mathbb{E}m_0^\lambda = e^{\psi(\lambda)}$

The measure change from $\mathbb{P}$ to $\mathbb{P}_\lambda$ corresponds to Cramér's change for the random walk $S_n = \sum_{i=0}^{n-1} \log m_i$ $(n \geq 1)$.

Proof of Cramér's expansion for $1 \le x = o(\sqrt{n})$ (2)

- Since $\log Z_n = S_n + \log W_n$, we have the decomposition:

$$\frac{\log Z_n - n\mu}{\sigma\sqrt{n}} = \frac{S_n - n\mu_\lambda}{\sigma\sqrt{n}} + \frac{\log W_n}{\sigma\sqrt{n}} + \frac{(\mu_\lambda - \mu)\sqrt{n}}{\sigma},$$

  where $\mu_\lambda = \mathbb{E}_\lambda \log m_0$.
- Choosing $\lambda > 0$ as solution of $x = \frac{(\mu_\lambda - \mu)\sqrt{n}}{\sigma}$, writing $Y_n^\lambda = \frac{S_n - n\mu_\lambda}{\sqrt{n}\sigma_\lambda}$ and $V_n^\lambda = \frac{\log W_n}{\sqrt{n}\sigma_\lambda}$ with σ_λ^2 = variance of $\log m_0$ under $\mathbb{P}_\lambda$, we obtain, with $\psi(\lambda) = \log \mathbb{E} e^{\lambda X} = \log \mathbb{E} m_0^\lambda$,

$$\begin{aligned} \mathbb{P}\left(\frac{\log Z_n - n\mu}{\sigma\sqrt{n}} > x\right) &= \mathbb{P}(Y_n^\lambda + V_n^\lambda > 0) && (4.6) \\ &= \mathbb{E}_\lambda\left[e^{(n\psi(\lambda) - \lambda S_n)} \mathbf{1}\{Y_n^\lambda + V_n^\lambda > 0\}\right] \\ &= \exp\left(n\psi(\lambda) - n\lambda\mu_\lambda\right) \times && (4.7) \\ &\quad \mathbb{E}_\lambda\left[e^{-\lambda\sigma_\lambda\sqrt{n}Y_n^\lambda} \mathbf{1}\{Y_n^\lambda + V_n^\lambda > 0\}\right] \end{aligned}$$

Proof of Cramér's expansion for $1 \leq x = o(\sqrt{n})$ (3)

- Using $Y_n^\lambda = \frac{S_n - n\mu_\lambda}{\sqrt{n}\sigma_\lambda} \to N(0,1)$ in law under $\mathbb{P}_\lambda$ and $V_n^\lambda = \frac{\log W_n}{\sqrt{n}\sigma_\lambda} \to 0$ to conclude. To make everything precise, we need:

1. Study the joint law of Y_n^λ and $\log W_n$ under $\mathbb{P}_\lambda$ uniformly for small λ
2. Use the fact that (Z_n) is still a supercritical branching process in a random environment satisfying Berry-Essen's bound.
3. Prove that under $\mathbb{P}_\lambda$, $\log W_n \to \log W$ in L^1 uniformly for small λ:

$$\sup_{0 \leq \lambda \leq \lambda_0} \mathbb{E}_\lambda |\log W - \log W_n|^p \leq C\delta_0^n, \quad \delta_0 < 1$$

4. Prove that W admits harmonic moments under $\mathbb{P}_\lambda$, uniformly for small λ: for some $a > 0$,

$$\sup_{0 \leq \lambda \leq \lambda_0} \mathbb{E}_\lambda e^{-tW} \leq Ct^{-a}$$

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Thank you !

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